ON A MODEL OF A HEREDITARILY ELASTIC SOLID DIFFERENTLY RESISTIVE TO TENSION AND COMPRESSION

PMM Vol. 35, №1, 1971, pp. 49-60
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Received May 19, 1970)

A theory for differently resistive hereditarily elastic solids is elucidated herein. The fundamental statements of different modulus elasticity theory [1–4], as well as the ordinary theory of hereditarily elastic solid [5–7], are utilized.

By analyzing the creep tests of various materials, it is easy to establish that in the enormous majority of cases they "creep differently", i.e. the strain processes under pure tension (plus) and pure compression (minus) proceed differently in time. It is also established by tests that these materials are of different modulus in many cases, i.e. the instantaneous moduli of elasticity in tension (E^+) and compression (E^-) also differ.

Attempts to construct a theory of a hereditarily elastic solid, differently resistive to tension and compression, have been made earlier [8–10]. These papers are not discussed at all herein since statements differing, in principle, from the initial statements of the theory proposed herein are underlying.

1. Let the material of the solid under consideration be such that under pure tension in any direction it has: An instantaneous modulus of elasticity E^+, an insatantaneous Poisson ratio ν^+, a hereditary strain coefficient $K^+(t-\tau)$ and a hereditary transverse strain coefficient $\mu^+(t-\tau)$; while for pure compression in any direction it has respectively: E^-, ν^-, $K^-(t-\tau)$, $\mu^-(t-\tau)$. As usual, t is here the time under consideration, and τ is the time when the stress would be applied, i.e. the age of the material up to loading time.

It is assumed that under simultaneous pure tension and compression in different mutually orthogonal directions, the characteristics of the solid remain invariant under uniform tension or compression.

It is assumed that the solid under consideration undergoes only slight strains for any state of stress, and is subject to the general regularities of a hereditary elastic continuum. In particular, according to the results elucidated in [2–4, 6], the existence of a creep potential is assumed for the three-dimensional state of stress, which as is known, is considered quite probable [6].

2. According to the Volterra principle of hereditary elasticity [6], the total strain of a solid consists of the instantaneous strain which is determined by the stress acting at this time, and the hereditary strain. Therefore, if a tensile stress $\sigma_{11}^\circ(\tau)$ acts for a time $d\tau$ at some point of a solid (the arguments x_i are omitted here and henceforth) at a time τ, then we shall have the instantaneous strains

$$e_{11}^\circ(t) = \sigma_{11}^\circ(t) / E^+, \quad e_{22}^\circ(t) = e_{33}^\circ(t) = -\nu^+\sigma_{11}^\circ(t) / E^+ \tag{2.1}$$

and the small hereditary strains

$$\begin{gathered} de_{11}^\circ = \sigma_{11}^\circ(\tau)\, d\tau K^+(t-\tau) \\ de_{22}^\circ = de_{33}^\circ = -\mu^+(t-\tau)\ \sigma_{11}^\circ(\tau) d\tau K^+(t-\tau) \end{gathered} \tag{2.2}$$

in the principal direction x_1° and the mutually perpendicular directions x_2° and x_3° orthogonal to x_1°.

If a compressive stress acts at the time τ then we shall evidently have, respectively

$$e_{11}^\circ(t) = -\sigma_{11}^\circ(t)/E^-, \qquad e_{22}^\circ(t) = e_{33}^\circ(t) = \nu^-\sigma_{11}^\circ(t)/E^- \tag{2.3}$$

$$de_{11}^\circ = -\sigma_{11}^\circ(\tau)\,d\tau K^-(t-\tau) \tag{2.4}$$

$$de_{22}^\circ = de_{33}^\circ = \mu^-(t-\tau)\sigma_{11}^\circ(\tau)\,d\tau K^-(t-\tau)$$

As usual [6], integrating the dependences (2.2) and (2.4) with respect to τ between $-\infty$ and t and adding the appropriate instantaneous elastic strains (2.1) and (2.3) thereto, we obtain the initial formulas of the law of hereditary elasticity.

For $\sigma_{11}^\circ(t) > 0$

$$e_{11}^\circ(t) = \frac{\sigma_{11}^\circ(t)}{E^+} + \int_{-\infty}^{t} \sigma_{11}^\circ(\tau)\,K^+(t-\tau)\,d\tau$$

$$e_{22}^\circ(t) = e_{33}^\circ(t) = -\nu^+\frac{\sigma_{11}^\circ(t)}{E^+} - \int_{-\infty}^{t} \mu^+(t-\tau)\,\sigma_{11}^\circ(\tau)\,K^+(t-\tau)\,d\tau \tag{2.5}$$

For $\sigma_{11}^\circ(t) < 0$

$$e_{11}^\circ(t) = -\frac{\sigma_{11}^\circ(t)}{E^-} - \int_{-\infty}^{t} \sigma_{11}^\circ(\tau)\,K^-(t-\tau)\,d\tau$$

$$e_{22}^\circ(t) = e_{33}^\circ(t) = \nu^-\frac{\sigma_{11}^\circ(t)}{E^-} + \int_{-\infty}^{t} \mu^-(t-\tau)\,\sigma_{11}^\circ(\tau)\,K^-(t-\tau)\,d\tau \tag{2.6}$$

3. Now, let there be a three-dimensional state of stress and let all the principal stresses $\sigma_{11}^\circ(t)$, $\sigma_{22}^\circ(t)$, $\sigma_{33}^\circ(t)$ act simultaneously at the point under consideration. In this case all the principal stresses are evidently either tensile $\sigma_{ii}^\circ(t) > 0$ or compressive $\sigma_{ii}^\circ(t) < 0$, or one of the principal stresses is of a different sign than the remaining two, i.e. two of the principal stresses are tensile and the third is compressive, or conversely, two of the principal stresses are compressive, and the third tensile. Clearly, any other general cases of the state of stress are impossible.

The first two cases, i.e. when $\sigma_{ii}^\circ(t) > 0$ or $\sigma_{ii}^\circ(t) < 0$, are of no special interest. In these cases, i.e. in domains and points of the first kind [1], the ordinary theory of a hereditary elastic solid holds with the appropriate strain coefficients. In particular, when $\sigma_{ii}^\circ(t) > 0$ we have E^+, ν^+, $\mu^+(t-\tau)$, $K^+(t-\tau)$ and when $\sigma_{ii}^\circ(t) < 0$ we have E^-, ν^-, $\mu^-(t-\tau)$, $K^-(t-\tau)$.

The remaining cases of a general state of stress will be considered in detail because these cases indeed characterize intrinsically the theory of hereditary elasticity of differently resistive solids. In these cases, i.e. in domains and points of the second kind [1], specific phenomena appear which are associated with the differing resistivity of the solid under consideration.

On the basis of the initial assumptions and relationships (2.5), (2.6), the hereditary elasticity law in the principal directions $x_i^\circ(t)$ will be written as follows:

$$e_{ii}^\circ(t) = (a_{ii} - a_{12})\sigma_{ii}^\circ(t) + a_{12}\sigma(t) + \int_{-\infty}^{t} \{[b_{ii}(t-\tau) - b_{12}(t-\tau)]\,\sigma_{ii}^\circ(\tau) + b_{12}(t-\tau)\,\sigma(\tau)\}\,d\tau \tag{3.1}$$

$$e_{ij}^{\circ} = 0 \quad (i \neq j), \qquad \sigma(t) = \sigma_{11}^{\circ}(t) + \sigma_{22}^{\circ}(t) + \sigma_{33}^{\circ}(t)$$

($i = 1, 2, 3$; $j = 1, 2, 3$; not summed over i)

Proceeding from the results presented above, particularly (2.5) and (2.6), we have for the coefficients a_{ik} and b_{ik} in different general cases of the state of stress

1) $\sigma_{11}^{\circ} > 0, \quad \sigma_{22}^{\circ} > 0, \quad \sigma_{33}^{\circ} > 0$

$$a_{11} = a_{22} = a_{33} = 1 / E^{+}, \quad a_{12} = -\nu^{+} / E^{+} \tag{3.2}$$

$$b_{11} = b_{22} = b_{33} = K^{+}(t-\tau), \quad b_{12} = -\mu^{+}(t-\tau)\, K^{+}(t-\tau)$$

2) $\sigma_{11}^{\circ} < 0, \quad \sigma_{22}^{\circ} < 0, \quad \sigma_{33}^{\circ} < 0$

$$a_{11} = a_{22} = a_{33} = 1 / E^{-}, \quad a_{12} = -\nu^{-} / E^{-} \tag{3.3}$$

$$b_{11} = b_{22} = b_{33} = K^{-}(t-\tau), \quad b_{12} = -\mu^{-}(t-\tau) K^{-}(t-\tau)$$

3) $\sigma_{11}^{\circ} > 0, \quad \sigma_{22}^{\circ} < 0, \quad \sigma_{33}^{\circ} > 0$

$$a_{11} = a_{33} = 1 / E^{+}, \quad a_{22} = 1 / E^{-}, \quad a_{12} = -\nu^{+} / E^{+} = -\nu^{-} / E^{-}$$

$$b_{11} = b_{33} = K^{+}(t-\tau), \qquad b_{22} = K^{-}(t-\tau) \tag{3.4}$$

$$b_{12} = -\mu^{+}(t-\tau) K^{+}(t-\tau) = -\mu^{-}(t-\tau)\, K^{-}(t-\tau)$$

4) $\sigma_{11}^{\circ} > 0, \quad \sigma_{22}^{\circ} < 0, \quad \sigma_{33}^{\circ} < 0$

$$a_{11} = 1 / E^{+}, \quad a_{22} = a_{33} = 1 / E^{-}, \quad a_{12} = -\nu^{+} / E^{+} = -\nu^{-} / E^{-}$$

$$b_{11} = K^{+}(t-\tau), \quad b_{22} = b_{33} = K^{-}(t-\tau) \tag{3.5}$$

$$b_{12} = -\mu^{+}(t-\tau) K^{+}(t-\tau) = -\mu^{-}(t-\tau)\, K^{-}(t-\tau) \quad \text{etc.}$$

The question of symmetry of the coefficients a_{ik} and b_{ik} is not discussed here in detail since by proceeding from the assumption of the existence of elastically instantaneous [4] and creep [6, 11] strain potentials, and by duplicating the whole reasoning expounded in [2–4, 6, 11], it can easily be shown that $a_{ik} = a_{ki}$ and $b_{ik} = b_{ki}$.

4. The law of hereditary elasticity can be found by proceeding from (3.1), for the initial orthogonal coordinate system x_i, relative to which the position of the principal directions x_i° of the point under consideration is defined by using the nine direction cosines l_i, m_i, n_i, which, written in tensor form, satisfy the following conditions:

$$c_k^{\,i} c_k^{\,j} = \delta_{ij} \tag{4.1}$$

where

$$\delta_{ij} = 0 \quad (i \neq j) \qquad \delta_{ij} = 1 \quad (i = j)$$

The functions c in (4.1) with elements of the coordinate systems x_i and x_i°, and also with the direction cosines, are connected by means of the known relationships [12]

$$c_i^{\,j} = \frac{\partial x_i^{\circ}}{\partial x_i} = \frac{\partial x_i}{\partial x_i^{\circ}}, \qquad c_i^{\,1} = \frac{\partial x_1^{\circ}}{\partial x_i} = \frac{\partial x_i}{\partial x_1^{\circ}} = l_i$$

$$c_i^{\,2} = \frac{\partial x_2^{\circ}}{\partial x_i} = \frac{\partial x_i}{\partial x_2^{\circ}} = m_i, \qquad c_i^{\,3} = \frac{\partial x_3^{\circ}}{\partial x_i} = \frac{\partial x_i}{\partial x_3^{\circ}} = n_i \tag{4.2}$$

According to (4.1) and (4.20), the formulas to transform the state of stress and strain from one orthogonal coordinate system x_i to another x_i° are written as follows [12]:

$$\sigma_{ij} = c_i^{\,p} c_j^{\,q} \sigma_{pq}^{\circ}, \qquad e_{ij} = c_i^{\,p} c_j^{\,q} e_{pq}^{\circ}$$

$$\sigma_{ij}^{\circ} = c_p^{\,i} c_q^{\,j} \sigma_{pq}, \qquad e_{ij}^{\circ} = c_p^{\,i} c_q^{\,j} e_{pq} \tag{4.3}$$

$$(p = 1, 2, 3; \; q = 1, 2, 3)$$

where, since the coordinate system x_i° coincides with the principal directions,

$$c_p{}^i c_q{}^j \sigma_{pq} = 0, \qquad c_p{}^i c_q{}^j e_{pq} = 0 \tag{4.4}$$

i.e. $\sigma_{ij}^\circ = 0$, $e_{ij}^\circ = 0$ for $i \neq j$.

Now, in the general case, let the state of stress and strain at a point of the second kind under consideration be characterized by the stresses $\sigma_{ij}(t)$ and the strains $e_{ij}(t)$ in the initial x_i coordinate system. The principal directions $x_i^\circ(t)$, the principal stresses $\sigma_{ii}^\circ(t)$ and the principal strains $e_{ii}^\circ(t)$ will correspond to this state of stress.

Under the simultaneous action of the principal stresses $\sigma_{ii}^\circ(t)$ at the time τ, the principal elastically instantaneous strains will appear, which are defined according to (3.1) by the formula

$$e_{ii}^\circ(\tau) = (a_{ii} - a_{12})\,\sigma_{ii}^\circ(\tau) + a_{12}\sigma(\tau) \quad \text{(not summed over } i\text{)} \tag{4.5}$$

and their corresponding elastically instantaneous strains in the initial x_i coordinate system will be defined according to (4.3) by using the formula

$$e_{ij}(\tau) = c_i{}^p(\tau)\, c_j{}^q(\tau)\, e^\circ_{pq}(\tau) \tag{4.6}$$

Now, let the principal stresses $\sigma_{ii}^\circ(\tau)$ act for a small time interval $d\tau$ while remaining invariant, then small hereditary strains will evidently be manifest in the principal directions $x_i^\circ(\tau)$, which will be defined according to (2.2), (2.4), (3.4) and (3.5) by

$$de_{ii}^\circ = \{[b_{ii}(t-\tau) - b_{12}(t-\tau)]\,\sigma_{ii}^\circ(\tau) + b_{12}(t-\tau)\,\sigma(\tau)\}\,d\tau \tag{4.7}$$

(not summed over i)

and according to (4.3) their corresponding small hereditary strains in the initial x_i coordinate system will be defined by

$$de_{ij} = c_i{}^p(\tau)\, c_j{}^q(\tau)\, de^\circ_{pq}(\tau) \tag{4.8}$$

Integrating (4.8) with respect to τ between the limits $-\infty$ and t, taking account of (4.7), and adding (4.6) to the corresponding elastically instantaneous strain obtained with (4.5) taken into account, we obtain three equivalent relationships after a number of manipulations [1], which represent the strains of a hereditary elastic solid in the initial coordinate system

$$\begin{aligned} e_{ij}(t) = {} & A_{11}\sigma_{ij}(t) + A_{12}\delta_{ij}\sigma(t) + C_3 m_i(t)\, m_j(t)\, \sigma_{22}^\circ(t) + C_2 n_i(t)\, n_j(t)\, \sigma_{33}^\circ(t) + \\ & + \int_{-\infty}^{t} [B_{11}(t-\tau)\,\sigma_{ij}(\tau) + B_{12}(t-\tau)\,\delta_{ij}\sigma(\tau) + D_3(t-\tau)\, m_i(\tau)\, m_j(\tau)\, \sigma_{22}^\circ(\tau) + \\ & + D_2(t-\tau)\, n_i(\tau)\, n_j(\tau)\, \sigma_{33}^\circ(\tau)]\, d\tau \end{aligned} \tag{4.9}$$

$$\begin{aligned} e_{ij}(t) = {} & A_{33}\sigma_{ij}(t) + A_{12}\delta_{ij}\sigma(t) - C_2 l_i(t)\, l_j(t)\, \sigma_{11}^\circ(t) - C_1 m_i(t)\, m_j(t)\, \sigma_{22}^\circ(t) + \\ & + \int_{-\infty}^{t} [B_{33}(t-\tau)\,\sigma_{ij}(\tau) + B_{12}(t-\tau)\,\delta_{ij}\sigma(\tau) - D_2(t-\tau)\, l_i(\tau)\, l_j(\tau)\, \sigma_{11}^\circ(\tau) - \\ & - D_1(t-\tau)\, m_i(\tau)\, m_j(\tau)\, \sigma_{22}^\circ(\tau)]\, d\tau \end{aligned} \tag{4.10}$$

$$\begin{aligned} e_{ij}(t) = {} & A_{22}\sigma_{ij}(t) + A_{12}\delta_{ij}\sigma(t) - C_3 l_i(t)\, l_j(t)\, \sigma_{11}^\circ(t) + C_1 n_i(t)\, n_j(t)\, \sigma_{33}^\circ(t) + \\ & + \int_{-\infty}^{t} [B_{22}(t-\tau)\,\sigma_{ij}(\tau) + B_{12}(t-\tau)\,\delta_{ij}\sigma(\tau) - D_3(t-\tau)\, l_i(\tau)\, l_j(\tau)\, \sigma_{11}^\circ(\tau) + \\ & + D_1(t-\tau)\, n_i(\tau)\, n_j(\tau)\, \sigma_{33}^\circ(\tau)]\, d\tau \end{aligned} \tag{4.11}$$

Here $i = 1, 2, 3$, $j = 1, 2, 3$, but there is no summation over j when $i = j$.

The representations (4.1)–(4.4) were utilized extensively in obtaining the elasticity relationships (4.9)–(4.11). New notation has been introduced in addition to that used earlier, namely:

$$
\begin{aligned}
&A_{ii} = a_{ii} - a_{12}, \qquad A_{12} = a_{12} \quad \text{(not summed over } i\text{)} \\
&C_1 = a_{33} - a_{22}, \qquad C_2 = a_{33} - a_{11}, \qquad C_3 = a_{22} - a_{11} \\
&B_{ii}(t-\tau) = b_{ii}(t-\tau) - b_{12}(t-\tau) \quad \text{(not summed over } i\text{)} \\
&B_{12}(t-\tau) = b_{12}(t-\tau), \qquad D_1(t-\tau) = b_{33}(t-\tau) - b_{22}(t-\tau) \\
&\qquad D_2(t-\tau) = b_{33}(t-\tau) - b_{11}(t-\tau) \\
&\qquad D_3(t-\tau) = b_{22}(t-\tau) - b_{11}(t-\tau)
\end{aligned}
\tag{4.12}
$$

Assuming

$$
\begin{gathered}
a_{11} = a_{22} = a_{33} = 1/E, \qquad a_{12} = -\nu/E \\
b_{11}(t-\tau) = b_{22}(t-\tau) = b_{33}(t-\tau) = K(t-\tau) \\
b_{12} = -\mu(t-\tau)K(t-\tau)
\end{gathered}
$$

in the elasticity relationships (4.9)–(4.11), we evidently arrive at the following elasticity relationship for a hereditary elastic solid:

$$
\begin{aligned}
e_{ij}(t) = \frac{\sigma_{ij}(t)(1+\nu) - \nu\delta_{ij}\sigma(t)}{E} + \int_{-\infty}^{t} \{K(t-\tau)[1+\mu(t-\tau)]\,\sigma_{ij}(\tau) - \\
- \mu(t-\tau)K(t-\tau)\,\delta_{ij}\sigma(\tau)\}\,d\tau
\end{aligned}
\tag{4.13}
$$

which agrees with the corresponding classical representation [5–7].

The representation (4.13) is even valid for domains of the first kind, it is hence necessary just to take into account that when $\sigma_{ii}^{\circ}(t) > 0$ we have E^+, ν^+, μ^+, K^+ and when $\sigma_{ii}^{\circ}(t) < 0$, we have E^-, ν^-, μ^-, K^-.

In general, in domains of the second kind the relationships (4.9)–(4.11) can be written in a rather shortened form if the signs of the principal stresses are known. The fact is that in the most general case at least two principal stresses have the same sign, whereupon (see (3.4), (3.5)) one of the coefficients C_i and one of the coefficients D_i become zero.

5. The elasticity relationships describing the shear strain occupy a special place in the mechanics of bodies of different resistivity.

Without restricting the generality of the discussion, let us examine shear phenomena in the $x_3 = 0$ plane for a plane state of stress ($\sigma_{33} = 0$, $\sigma_{23} = 0$, $\sigma_{13} = 0$).

Let us assume

$$
x_1 = x, \quad x_2 = y, \quad \sigma_{12} = \tau_{xy}, \quad e_{12} = e_{xy}/2, \quad \sigma_{33}^{\circ} = \sigma_{33} = 0,
$$

then we obtain for the shear strain from (4.9)

$$
\begin{aligned}
e_{xy}(t) = 2A_1\tau_{xy}(t) + 2C_3 m_1(t)\,m_2(t)\,\sigma_{22}^{\circ}(t) + \\
+ 2\int_{-\infty}^{t} [B_1(t-\tau)\,\tau_{xy}(\tau) + D_3(t-\tau)\,m_1(\tau)\,m_2(\tau)\,\sigma_{22}^{\circ}(\tau)]\,d\tau
\end{aligned}
\tag{5.1}
$$

Now, if we formally assume [5]

$$e_{xy}(t) = \frac{1}{G'}\tau_{xy}(t) + \int_{-\infty}^{t} \omega'(t-\tau)\,\tau_{xy}(\tau)\,d\tau \tag{5.2}$$

then we obtain for the formally introduced concepts of the shear modulus G' and the coefficient of hereditary shear strain $\omega'(t-\tau)$

$$\frac{1}{G'} = 2\left[A_1 \frac{\sigma_{11}^\circ(t)}{\sigma_{11}^\circ(t) - \sigma_{22}^\circ(t)} - A_2 \frac{\sigma_{22}^\circ(t)}{\sigma_{11}^\circ(t) - \sigma_{22}^\circ(t)}\right] \tag{5.3}$$

$$\omega'(t-\tau) = 2\left[B_1(t-\tau)\frac{\sigma_{11}^\circ(\tau)}{\sigma_{11}^\circ(\tau) - \sigma_{22}^\circ(\tau)} - B_2(t-\tau)\frac{\sigma_{22}^\circ(\tau)}{\sigma_{11}^\circ(\tau) - \sigma_{22}^\circ(\tau)}\right] \tag{5.4}$$

Therefore, the quantities G' and $\omega'(t-\tau)$, provisionally called the shear modulus and the coefficient of hereditary shear strain, respectively, depend essentially on the state of stress of the considered point of the solid [1].

It is clear from the above that attempts to construct a general theory of the deformation of solids of differing resistivity under the assumption that the shear strains can be described by relationships such as (5.2) by the introduction of some values for G' and ω' found from pure shear experiments cannot be considered correct.

Let us examine the pure shear phenomenon. Let $\sigma_x(t) = 0$, $\sigma_y(t) = 0$, $\tau_{xy}(t) = p(t)$. Then, evidently

$$\sigma_{11}^\circ(t) = p(t), \qquad \sigma_{22}^\circ(t) = -p(t)$$

Taking the above into account, we obtain for the pure shear strain from (5.2)–(5.4) according to (4.1)

$$e_{xy}(t) = (a_{11} + a_{22} - 2a_{12})\,p(t) + \int_{-\infty}^{t} [b_{11}(t-\tau) + b_{22}(t-\tau) - 2b_{12}(t-\tau)]\,p(t)\,d\tau \tag{5.5}$$

Since $\sigma_{11}^\circ(t) > 0$, $\sigma_{22}^\circ(t) < 0$, we have for the coefficients a_{ik} and b_{ik}

$$\begin{aligned} &a_{11} = 1/E^+, \quad a_{22} = 1/E^-, \quad a_{12} = -\nu^+/E^+ = -\nu^-/E^- \\ &b_{11}(t-\tau) = K^+(t-\tau), \quad b_{22}(t-\tau) = K^-(t-\tau) \\ &b_{12}(t-\tau) = -\mu^+(t-\tau)K^+(t-\tau) = -\mu^-(t-\tau)K^-(t-\tau) \end{aligned} \tag{5.6}$$

Substituting the values of a_{ik} and b_{ik} from (5.6) into (5.5), we obtain

$$e_{xy}(t) = \left(\frac{1+\nu^+}{E^+} + \frac{1+\nu^-}{E^-}\right)p(t) + \\ + \int_{-\infty}^{t}\{K^+(t-\tau)[1+\mu^+(t-\tau)] + K^-(t-\tau)[1+\mu^-(t-\tau)]\}\,p(\tau)\,d\tau \tag{5.7}$$

We hence obtain for the shear modulus G' in pure shear and for the hereditary strain coefficient in pure shear ω'

$$G' = E^+E^- / [E^+(1+\nu^-) + E^-(1+\nu^+)]$$

$$\omega' = K^+(t-\tau)[1+\mu^+(t-\tau)] + K^-(t-\tau)[1+\mu^-(t-\tau)] \tag{5.8}$$

Let us assume

$$E^+ = E^- = E,\ \nu^+ = \nu^- = \nu, \quad K^+(t-\tau) = K^-(t-\tau) = \\ = K(t-\tau), \quad \mu^+(t-\tau) = \mu^-(t-\tau) = \mu(t-\tau),$$

then from (5.8) we obtain the values of the shear modulus G and the coefficient of hereditary strain in pure shear $\omega(t-\tau)$ for a classical hereditary elastic solid

$$G = E / 2(1+\nu), \qquad \omega = 2[1+\mu(t-\tau)]K(t-\tau) \tag{5.9}$$

which are in agreement with known results from classical theory [5].

By virtue of (3.4), (3.5), the value of $\omega'(t-\tau)$ can be represented by different formulas also in the general case, namely:

$$\omega'(t-\tau) = K^{+}(t-\tau)[1+2\mu^{+}(t-\tau)] + K^{-}(t-\tau)$$
$$\omega'(t-\tau) = K^{-}(t-\tau)[1+2\mu^{-}(t-\tau)] + K^{+}(t-\tau)$$

6. Substituting the values of A_{ij} and B_{ij} into the relationship (4.9) (the other equivalent relationships (4.10) and (4.11) will not be examined at all), and performing the appropriate summation ($e(t) = e_{ii}(t)$), we obtain for the volume strain

$$e(t) = (a_{11}+2a_{12})\sigma(t) + C_3\sigma_{22}^{\circ}(t) + C_2\sigma_{33}^{\circ}(t) + \int_{-\infty}^{t}\{[b_{11}(t-\tau)+2b_{12}(t-\tau)]\sigma(\tau) + D_3(t-\tau)\sigma_{22}^{\circ}(\tau) + D_2(t-\tau)\sigma_{33}^{\circ}(\tau)\}\,d\tau \tag{6.1}$$

where, as usual, $\sigma(t) = \sigma_{ii}^{\circ}(t) = \sigma_{ii}(t)$, or according to (4.12)

$$e(t) = (a_{11}+2a_{12})\sigma_{11}^{\circ}(t) + (a_{22}+2a_{12})\sigma_{22}^{\circ}(t) + (a_{33}+2a_{12})\sigma_{33}^{\circ}(t) + \int_{-\infty}^{t}\{[b_{11}(t-\tau)+2b_{12}(t-\tau)]\sigma_{11}^{\circ}(\tau) + \tag{6.2}$$
$$+ [b_{22}(t-\tau)+2b_{12}(t-\tau)]\sigma_{22}^{\circ}(\tau) + [b_{33}(t-\tau)+2b_{12}(t-\tau)]\sigma_{33}^{\circ}(\tau)\}\,d\tau$$

Solving the integral equation (6.1) for the function $\sigma(t)$, we obtain

$$\sigma(t) = \frac{e(t)-f_1(t)}{a_{11}+2a_{12}} - \int_{-\infty}^{t}\Gamma_1(t-\tau)[e(\tau)-f_1(\tau)]\,d\tau \tag{6.3}$$

$$f_1(t) = C_3\sigma_{22}^{\circ}(t) + C_2\sigma_{33}^{\circ}(t) + \int_{-\infty}^{t}[D_3(t-\tau)\sigma_{22}^{\circ}(\tau) + D_2(t-\tau)\sigma_{33}^{\circ}(\tau)]\,d\tau \tag{6.4}$$

where $\Gamma_1(t-\tau)$ is the resolvent of the kernel of the integral equation, and the function $f_1(t)$ characterizes the different resistivity of the considered solid.

Subtracting one third of the volume strain (6.1), we obtain from (4.9)

$$e_{ii}(t) - {}^1/_3\, e(t) = (a_{11}-a_{12})[\sigma_{ii}(t) - {}^1/_3\,\sigma(t)] + \int_{-\infty}^{t}[b_{11}(t-\tau)-b_{12}(t-\tau)][\sigma_{ii}(\tau) - {}^1/_3\,\sigma(\tau)]\,d\tau + f_2(t) \tag{6.5}$$

(not summed over i)

$$f_2(t) = C_3\left[m_i^2(t)-\frac{1}{3}\right]\sigma_{22}^{\circ}(t) + C_2\left[n_i^2(t)-\frac{1}{3}\right]\sigma_{33}^{\circ}(t) + \int_{-\infty}^{t}\left\{D_3(t-\tau)\left[m_i^2(\tau)-\frac{1}{3}\right]\sigma_{22}^{\circ}(\tau) + D_2(t-\tau)\left[n_i^2(\tau)-\frac{1}{3}\right]\sigma_{33}^{\circ}(\tau)\right\}d\tau \tag{6.6}$$

Solving the integral equation (6. 5) for the stresses $\sigma_{ii}(t)$ while taking account of (6. 3), we obtain (not summed over i)

$$\sigma_{ii}(t) = \frac{e_{ii}(t) - {}^1/_3\, e(t) - f_2(t)}{a_{11} - a_{12}} - \int_{-\infty}^{t} \Gamma_2(t-\tau)\Big[e_{ii}(\tau) - \\ - \frac{1}{3} e(\tau) - f_2(\tau)\Big] d\tau + \frac{e(t) - f_1(t)}{3(a_{11} + 2a_{12})} - \\ - \frac{1}{3} \int_{-\infty}^{t} \Gamma_1(t-\tau)\,[e(\tau) - f_1(\tau)]\, d\tau \tag{6.7}$$

where $\Gamma_2(t-\tau)$ is the resolvent of the kernel of the integral equation (6. 5) with the kernel $[b_{11}(t-\tau) - b_{12}(t-\tau)]$.

Solving the integral equation (4. 9) for the stresses $\sigma_{ij}(t)$, we obtain (when $i \neq j$)

$$\sigma_{ij}(t) = \frac{e_{ij}(t) - f_3(t)}{a_{11} - a_{12}} - \int_{-\infty}^{t} \Gamma_2(t-\tau)\,[e_{ij}(\tau) - f_3(\tau)]\, d\tau \tag{6.8}$$

$$f_3(t) = C_3 m_i(t)\, m_j(t)\, \sigma_{22}^{\circ}(t) + C_2 n_i(t)\, n_j(t)\, \sigma_{33}^{\circ}(t) + \\ + \int_{-\infty}^{t} [D_3(t-\tau)\, m_i(\tau)\, m_j(\tau)\, \sigma_{22}^{\circ}(\tau) + D_2(t-\tau)\, n_i(\tau)\, n_j(\tau)\, \sigma_{33}^{\circ}(\tau)]\,d\tau \tag{6.9}$$

The representations $f_i(t)$, which characterize the differing resistivity of the considered solid, and are zero in particular for a classical elastically hereditary body ($C_3 = C_2 = 0$, $D_3(t-\tau) = D_2(t-\tau) = 0$) , are contained in all the formulas presented above for the stresses (6. 3), (6. 7), (6. 8). The principal stresses $\sigma_{22}^0(t)$, $\sigma_{33}^0(t)$ and the direction cosines of the principal directions which can be represented in terms of the strains only after the solution of a system of integral equations (3. 1) and the joint solution of the system (4. 1), (4. 3), (4. 4) for the direction cosines, enter into these representations. In the general case this procedure is quite tedious and there is no opportunity to present it here. These question have been discussed in [1–4] for the particular case of a body of different moduli.

7. To solve the problems of a hereditary elastic solid in terms of stresses, let us proceed from the strain continuity equations which are [12]

$$e_{is,jk} + e_{jk,is} - e_{ik,js} - e_{js,ik} = 0 \tag{7.1}$$

Let us introduce the notation for the linear operators [7]

$$L_{sp}(q) = A_{sp} q(t) + \int_{-\infty}^{t} B_{sp}(t-\tau)\, q(\tau)\, d\tau$$

$$N_k[q_i q_j \sigma_{rr}^{\circ}] = C_k q_i(t)\, q_j(t)\, \sigma_{rr}^{\circ}(t) + \int_{-\infty}^{t} D_k(t-\tau)\, q_i(\tau)\, q_j(\tau)\, \sigma_{rr}^{\circ}(\tau)\, d\tau \tag{7.2}$$

According to (7. 2), the expressions (4. 9) for the strain are rewritten as follows:

$$e_{ij}(t) = L_{11}(\sigma_{ij}) + \delta_{ij} L_{13}(\sigma) + N_3[m_i m_j \sigma_{22}^{\circ}] + N_2[n_i n_j \sigma_{33}^{\circ}] \tag{7.3}$$

Substituting the values of e_{ij} from (7. 3) into (7. 1), we obtain the following system of six equations in the desired stresses σ_{ij}

$$\begin{aligned}&L_{11}(\sigma_{is,jk}+\sigma_{jk,is}-\sigma_{ik,js}-\sigma_{js,ik})+L_{12}(\delta_{is}\sigma_{,jk}+\\&+\delta_{jk}\sigma_{,is}-\delta_{ik}\sigma_{,js}-\delta_{js}\sigma_{,ik})+N_5[(m_im_s\sigma_{22}^\circ)_{,jk}+\\&+(m_jm_k\sigma_{22}^\circ)_{,is}-(m_im_k\sigma_{22}^\circ)_{,js}-(m_jm_s\sigma_{22}^\circ)_{,ik}]+\\&+N_2[(n_in_s\sigma_{33}^\circ)_{,jk}+(n_jn_k\sigma_{33}^\circ)_{,is}-(n_in_k\sigma_{33}^\circ)_{,is}-(n_jn_s\sigma_{33}^\circ)_{,ik}]=0\end{aligned} \quad (7.4)$$

Boundary conditions, having the customary form [12]

$$X_i^* = \sigma_{ij}n^i \quad (7.5)$$

should be added to the system of equations (7.4), where X_i^* are the surface loading components, and n^i the components of the unit normal to the surface.

The solution of the system of integro-differential equations (7.4) is fraught with great difficulties in the general case. However, as is easy to note, when there are no shear stresses in the initial x_i coordinate system in domains of the first kind, the system of equations (7.4) simplifies and takes on the structure of the corresponding classical equations.

8. The initial equations of the theory of a hereditary elastic solid in displacements can be written by starting from the equilibrium equations which are in the initial coordinate system [12]

$$\sigma_{ij,j} + \rho X_i = 0 \quad (8.1)$$

where ρX_i are the volume force components.

Let us introduce the following linear operators [7]:

$$R(q) = \frac{q(t)}{a_{11}-a_{12}} - \int_{-\infty}^{t} \Gamma_2(t-\tau)\,q(\tau)\,d\tau$$
$$Q(q) = \frac{q(t)}{3(a_{11}+2a_{12})} - \frac{1}{3}\int_{-\infty}^{t} \Gamma_1(t-\tau)\,q(\tau)\,d\tau \quad (8.2)$$

then according to (6.7), (6.8), we obtain for the stresses σ_{ij}

$$\sigma_{ij}(t) = R(e_{ij} - {}^1/_3\delta_{ij}e) + \delta_{ij}Q(e) - R[\delta_{ij}f_2 + (1-\delta_{ij})f_3] - \delta_{ij}Q(f_1) \quad (8.3)$$

Substituting the values of σ_{ij} from (8.3) into the equilibrium equation, and taking into account that

$$e_{ij} = {}^1/_2(u_{i,j}+u_{j,i}), \qquad e = u_{k,k}, \qquad \delta_{ij}u_{k,kj} = u_{j,ij} \quad (8.4)$$

we obtain the following system of integro-differential equations in the desired displacements:

$$\begin{aligned}&R({}^1/_2u_{i,jj} + {}^1/_6u_{j,ij}) + Q(u_{j,ij}) - R[\delta_{ij}f_{2,j} +\\&+(1-\delta_{ij})f_{3,j}] - Q(\delta_{ij}f_{1,j}) + \rho X_i = 0\end{aligned} \quad (8.5)$$

Eliminating terms containing quantities which characterize the different resistivities of the solid from the system (8.5), i.e. terms with f_i, we obtain the corresponding equations of the ordinary theory.

The solution of the system (8.5), just as (7.4), is a complex problem even in the case of classical theory. However, they can be solved in some particular cases even in the case of a solid of different resistivity.

9. Let the mechanical characteristics of the solid under consideration be such that the following dependencies exist between the linear operators of a hereditary elastic solid:

$$L_{sp}(q) = \frac{A_{sp}}{C_k} N_k [q_i q_j \sigma_{rr}^\circ], \qquad R(q) = \frac{3(a_{11} + 2a_{12})}{a_{11} - a_{12}} Q(q) \tag{9.1}$$

whereupon we easily obtain in conformity with (8.2) and (9.1)

$$L_{sp}(q) = A_{sp} M_{sp}(q), \qquad R(q) = \frac{P(q)}{a_{11} + a_{12}} \tag{9.2}$$

$$N_k [q_i q_j \sigma_{rr}^\circ] = C_k M_k(q), \qquad Q(q) = \frac{P(q)}{3(a_{11} + 2a_{12})}$$

where

$$M(q) = M_{sp}(q) = M_k(q) = q(t) + \frac{1}{A_{sp}} \int_{-\infty}^{t} B_{sp}(t - \tau) q(\tau) d\tau =$$

$$= q_i(t) q_j(t) \sigma_{rr}^\circ(t) + \frac{1}{C_k} \int_{-\infty}^{t} D_k(t - \tau) q_i(\tau) q_j(\tau) \sigma_{rr}^\circ(\tau) d\tau$$

$$P(q) = q(t) - (a_{11} - a_{12}) \int_{-\infty}^{t} \Gamma_2(t - \tau) q(\tau) d\tau = \tag{9.3}$$

$$= q(t) - (a_{11} + 2a_{12}) \int_{-\infty}^{t} \Gamma_1(t - \tau) q(\tau) d\tau$$

Taking account of (9.1), (9.2), the governing systems of equations (7.4) and (7.5) are rewritten as follows:

$$M \{A_{11} (\sigma_{is,\,jk} + \sigma_{jk,\,is} - \sigma_{ik,\,js} - \sigma_{js,\,ik}) +$$

$$+ A_{12} (\delta_{is}\sigma_{,\,jk} + \delta_{jk}\sigma_{,\,is} - \delta_{ik}\sigma_{,\,js} - \delta_{js}\sigma_{,\,ik}) + C_3 [(m_i m_j \sigma_{22}^\circ)_{,\,jk} +$$

$$+ (m_j m_k \sigma_{22}^\circ)_{,\,is} - (m_i m_k \sigma_{22}^\circ)_{,\,is} - (m_j m_s \sigma_{22}^\circ)_{,\,ik}] +$$

$$+ C_2 [(n_i n_s \sigma_{33}^\circ)_{,\,jk} + (n_j n_k \sigma_{33}^\circ)_{,\,is} - (n_i n_k \sigma_{33}^\circ)_{,\,is} - (n_j n_s \sigma_{33}^\circ)_{,\,ik}]\} = 0 \tag{9.4}$$

and finally

$$P \left\{ \frac{1}{a_{11} - a_{12}} \left(\frac{1}{2} u_{i,\,jj} + \frac{1}{6} u_{j,\,ij} \right) + \frac{1}{3(a_{11} + 2a_{12})} (u_{j,\,ij}) - \right.$$

$$- \frac{1}{a_{11} - a_{12}} [\delta_{ij} f_{2,\,j} + (1 - \delta_{ij}) f_{3,\,j}] -$$

$$\left. - \frac{1}{3(a_{11} + 2a_{12})} (\delta_{ij} f_{1,\,j}) \right\} + \rho X_i = 0 \tag{9.5}$$

Examining the system of equations (9.4) and (9.5), let us note that the corresponding equations of different-modulus elasticity theory [1] are written in the braces in both systems of equations. The linear operators $M(q)$ and $P(q)$, which characterize the hereditary elastic properties of the considered solid, are the multipliers of these equations.

Therefore, if the conditions (9.1) hold in domains of the second kind of a solid of different resistivity, then methods of different-modulus theory of elasticity can be utilized in considering the state of stress and strain. This means that the Arutiunian theorems will be valid for a certain class of problems [5, 7] when conditions (9.1) are conserved, even for solids of different resistivity. As regards the domains of the first kind, the initial equation will here have the customary structure with appropriate mechanical characteristics (E^+, ν^+, μ^+, K^+ or E^-, ν^-, μ^-, K^-), and the Arutiunian theorems will be acceptable in the classical formulation [5].

10. Questions of approximating experimental strain curves by using some function of the time [5−7] occupy a special place in researches on the mechanics of hereditary elastic solids.

This question is not discussed here since known methods appied for this purpose in the ordinary theory [5−7] are completely acceptable even in the case of solids of different resistivity. Appropriate analytical representations can be found in the customary manner when time strain curves in pure tension and compression are available [13−16].

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Translated by M. D. F.